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# Gaussian Measures on a Banach Space

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## 1. INTRODUCTION

The equivalence and singularity of Gaussian measures is a problem which has been the subject of a great deal of study. The papers by Feldman, Gikhman and Skorkhod, Kallianpur and Oodaira, Parzen, Segal, Shepp, and Varberg indicated in the bibliography are ample evidence of this fact. In addition, there is the work of Cameron and Martin which was the first of this type. Here we will show that using similar techniques one is able to handle Gaussian measures on any real separable Banach space, and that these results easily apply to Gaussian processes. In particular, it is the results of [8] and [13] that are critical here.

Throughout this paper $B$ will denote a real separable Banach space with norm $\|\cdot\|_B$ and the minimal sigma-algebra containing the open sets of $B$, called the Borel sets, will be denoted by $\mathcal{O}_B$. Our first objective is to define a particular inner product on $B$ which generates a norm weaker than $\|\cdot\|_B$. Upon completing $B$ with respect to this norm we will obtain a separable Hilbert space $\tilde{H}$ with the prescribed inner product. Since the norm on $\tilde{H}$ given by the inner product is weaker than $\|\cdot\|_B$ on $B$ it follows that $B$ intersected with the Borel subsets of $\tilde{H}$, denoted by $\mathcal{O}_{\tilde{H}}$, is a subsigma-algebra of $\mathcal{O}_B$. In fact, we will choose the inner product so that $B \cap \mathcal{O}_{\tilde{H}} = \mathcal{O}_B$. Thus, any measure $\mu$ on $\mathcal{O}_B$ immediately induces a measure on $(\tilde{H}, \mathcal{O}_{\tilde{H}})$ by defining subsets of $\tilde{H} - B$ to be of measure zero. In the case that $\mu$ is Gaussian we will show that by investigating $\mu$ on $\tilde{H}$ we are able to obtain a great deal of information about $\mu$ on $B$. In particular, we are able to provide necessary and sufficient conditions that two Gaussian measures on $B$ are equivalent. As an application of this

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result we obtain an easy proof of Theorem 1 of Shepp [14] regarding the equivalence of a Gaussian process to the Wiener process, and also a result similar to those of Kallianpur and Oodaira [9]. It is also possible to apply our results to certain Gaussian measures on the $\ell_p$, $1 \leqslant p < \infty$, spaces [15], [16].

2. We first prove a lemma obtaining the inner product on $B$ which will be central to our further work.

LEMMA 2.1. *If $B$ is a real separable Banach space with norm $\|\cdot\|_B$, then there exists an inner product $(\ ,\ )$ on $B$ such that the norm generated by $(\ ,\ )$ is weaker than $\|\cdot\|_B$, and if $\bar{H}$ is the completion of $B$ under the inner product norm, then $\mathcal{O}_B \subseteq \mathcal{O}_{\bar{H}}$ so $\mathcal{O}_B = B \cap \mathcal{O}_{\bar{H}}$.*

*Proof.* If $B$ is finite dimensional the result is obvious so assume $B$ is infinite dimensional. Let $x_1, x_2, \ldots$ be a dense subset of $B$ and let $F_n$ be a bounded linear functional on $B$ such that $\|F_n\| = 1$ and $F_n(x_n) = \|x_n\|_B$. Let $\{t_n\}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} t_n = 1$. If $x, y \in B$ we define

$$(x, y) = \sum_{n=1}^{\infty} t_n F_n(x) F_n(y).$$

Then $(\ ,\ )$ is an inner product on $B$ and

$$(x, x) = \sum_{n=1}^{\infty} t_n[F_n(x)]^2 \leqslant \sup_n [F_n(x)]^2 = \|x\|_B^2.$$

Let $\bar{H}$ denote the completion of $B$ under $(\ ,\ )$ and let $\|\cdot\|_{\bar{H}}$ denote the norm on $\bar{H}$ induced by the inner product $(\ ,\ )$. Then $\|x\|_{\bar{H}} \leqslant \|x\|_B$ for $x \in B$.

We now show $\mathcal{O}_B \subseteq \mathcal{O}_{\bar{H}}$. Since $\|\cdot\|_{\bar{H}}$ is weaker than $\|\cdot\|_B$ on $B$ it follows that $\mathcal{O}_{\bar{H}} \cap B \subseteq \mathcal{O}_B$ and hence $\mathcal{O}_B \subseteq \mathcal{O}_{\bar{H}}$ actually implies $\mathcal{O}_{\bar{H}} \cap B = \mathcal{O}_B$. First we observe that for $x \in B$ we have $\|x\|_B = \sup_n |F_n(x)|$ and that $\bar{H}$ is isometric to a closed subspace of the Hilbert space $\mathscr{H}$ of all sequences of real numbers $\{x_n\}$ such that $\sum_{k=1}^{\infty} t_k x_k^2 < \infty$ with inner product $(\{x_k\}, \{y_k\}) = \sum_{k=1}^{\infty} t_k x_k y_k$. That is, since $\sum_{k=1}^{\infty} t_k = 1$ it follows that the set of all bounded sequences, denoted by $\ell_\infty$, is a subspace of $\mathscr{H}$. Here $\|\cdot\|_\infty$ denotes the usual $\ell_\infty$ norm and if $\phi(x) = \{F_k(x)\}$, $x \in B$, then $\phi$ is an isometry of $B$ into $\ell_\infty$. Furthermore, $\phi$ extended from $B$ to $\bar{H}$ maps $\bar{H}$ isometrically into a closed subspace $D$ of $\mathscr{H}$ as stated.

Let $G = \phi(B)$. We will show that $G \in \mathcal{O}_{\mathscr{H}}$ and that any $\|\cdot\|_\infty$ open subset of $G$ is also in $\mathcal{O}_{\mathscr{H}}$. Thus $\mathcal{O}_G$ is a sigma algebra in $\mathcal{O}_{\mathscr{H}}$. Since

$D$ is a closed subspace of $\mathscr{H}$ the Borel subsets of $D$, call them $\mathscr{F}$, form a sigma algebra of $\mathcal{O}_{\mathscr{H}}$ and $\mathscr{F} = D \cap \mathcal{O}_{\mathscr{H}}$. Thus $\mathcal{O}_G \subseteq \mathscr{F}$ since $\mathcal{O}_G \subseteq \mathcal{O}_{\mathscr{H}}$ and $G \subseteq D$. By isometry we then find $\mathcal{O}_B \subseteq \mathcal{O}_{\tilde{H}}$ as desired. We now turn to the proof that $\mathcal{O}_G \subseteq \mathcal{O}_{\mathscr{H}}$.

Let $I_j(N) = \{y : \| y - x_j \|_\infty < 1/N\}$ for $j = 1, 2,...; N = 1, 2,...$ where $x_1, x_2,...$ are a dense subset of $G$. Then $I_j(N) \in \mathcal{O}_{\mathscr{H}}$ since $I_j(N) = \bigcap_{i=1}^{\infty} \{y : | y^i - x_j^i | < 1/N\}$ where $y^i$ and $x_j^i$ denote the $i$-th coordinate of $y$ and $x_j$, respectively. Now $G = \bigcap_{N=1}^{\infty} \bigcup_{j=1}^{\infty} I_j(N)$ so $G \in \mathcal{O}_{\mathscr{H}}$. Now suppose $V$ is an open set in $G$. Then

$$V = \bigcup_{r=1}^{\infty} \{y \in G : \| y - z_r \|_\infty < \epsilon_r\}$$

where $z_1, z_2,...$ are in $V \subseteq G$. However,

$$\{y \in G : \| y - z_r \|_\infty < \epsilon_r\} = G \cap \{y : \| y - x_r \|_\infty < \epsilon_r\}$$
$$= G \cap \bigcap_{i=1}^{\infty} \{y : | y^i - x_r^i | < \epsilon_r\}$$

so $V \in \mathcal{O}_{\mathscr{H}}$. Thus open sets in $G$ are in $\mathcal{O}_{\mathscr{H}}$ and hence $\mathcal{O}_G \subseteq \mathcal{O}_{\mathscr{H}}$ so the proof is complete.

A measure $\mu$ on $\mathcal{O}_B$ is defined to be *Gaussian* if, for every linear functional $T$ on $B$, $T(x)$ has a Gaussian distribution and there exists a vector $a_\mu$ in $B$ such that $T(a_\mu)$ is the mean of $T(x)$.

The next lemma follows immediately.

LEMMA 2.2. *If $\mu$ is a Gaussian measure on $(B, \mathcal{O}_B)$ and $(\tilde{H}, \mathcal{O}_{\tilde{H}})$ is defined as in Lemma 2.1 then $\mu(A) = \mu(A \cap B)$, $A \in \mathcal{O}_{\tilde{H}}$, extends $\mu$ to be a Gaussian measure on $(\tilde{H}, \mathcal{O}_{\tilde{H}})$.*

Throughout the remainder of the paper we will assume that $\tilde{H}$ is derived from $B$ as in Lemma 2.1 and that any measure $\mu$ on $(B, \mathcal{O}_B)$ is also defined on $(\tilde{H}, \mathcal{O}_{\tilde{H}})$ as in Lemma 2.2.

Now if $\mu$ is a Gaussian measure on $B$ and hence on $\tilde{H}$ it follows that there exists a nonnegative symmetric trace class operator $A_\mu$ on $\tilde{H}$ (such an operator is often called an $S$-operator) such that $(A_\mu x, x)_{\tilde{H}} = \int_{\tilde{H}} (x, y - a_\mu)^2 \, d\mu(y)$ for $x \in \tilde{H}$ and that $\mu$ is uniquely determined on $\tilde{H}$ by the operator $A_\mu$ and the mean $a_\mu$. These results are well-known and appear, for example, in Sazonov [12]. Further, if $A_\mu$ is an $S$-operator on $\tilde{H}$ it is known that

$$A_\mu(\cdot) = \sum_k \lambda_k(\cdot, \phi_k)_{\tilde{H}} \phi_k \tag{2.1}$$

on $\tilde{H}$ where $\{\phi_k\}$ is an orthonormal sequence in $\tilde{H}$ and $\lambda_k > 0$, $\sum_k \lambda_k < \infty$. Consequently, given $A_\mu$ we are able to define the Hilbert space

$$H = \left\{x \in \tilde{H} : x \in \operatorname{span}\{\phi_1, \phi_2, \ldots\}, \sum_k \frac{(x, \phi_k)_{\tilde{H}}^2}{\lambda_k} < \infty\right\} \tag{2.2}$$

where for $x, y \in H$ the inner product is

$$(x, y)_H = \sum_k \frac{(x, \phi_k)_{\tilde{H}} (y, \phi_k)_{\tilde{H}}}{\lambda_k}. \tag{2.3}$$

We say a measure $\mu$ is *quasi-invariant* under translation by the vector $a$ if $\mu(E) = 0$ implies $\mu(E + a) = 0$.

LEMMA 2.3. *If $\mu$ is a Gaussian measure on $B$ which induces the $S$-operator $A_\mu$ on $\tilde{H}$ of the form* (2.1) *and $H$ is defined as in* (2.2), *then*

(1) $H \subseteq B \subseteq \tilde{H}$

(2) *$\mu$ is quasi-invariant under translation by the vector $a$ iff $a - a_\mu$ is in $H$.*

(3) *If $\mu^*(E) = \mu(E + a_\mu)$ then $\mu^*$ is the restriction to $(B, \mathcal{O}\!\ell_B)$ of the Gaussian measure on $\tilde{H}$ induced by the canonical normal distribution defined on $H$.*

*Proof.* When $\mu$ is viewed as a Gaussian measure on $(\tilde{H}, \mathcal{O}\!\ell_{\tilde{H}})$, then (2) follows from, for example, Theorem 4.1 of [7]. Clearly, $H \subseteq \tilde{H}$ and $B \subseteq \tilde{H}$. If $H \not\subseteq B$ then by (2) there exists $a \in H - B$ such that $\mu$ is quasi-invariant under translation by $a + a_\mu$, and $B \cap (B + a) = \phi$ which implies $1 = \mu(\tilde{H}) \geqslant \mu(B) + \mu(B + a) = 2$ since $\mu(B) = 1$. Hence $H \subseteq B$ and (1) follows. To prove (3) simply observe that $\|\cdot\|_{\tilde{H}}$ is a measurable norm on $H$ (see Gross [8]), that $(A_\mu x, x)_{\tilde{H}} = (A_\mu x, A_\mu x)_H$ for every $x \in \tilde{H}$, and apply Theorem 1 of [8] since the linear functional $T(\cdot) = (\cdot, x)_{\tilde{H}}$ restricted to $H$ is $(\cdot, A_\mu x)_H$.

3. THEOREM 3.1. *If $\mu$ and $\nu$ are Gaussian measures on $B$, if $\tilde{H}$ is as defined for Lemma* 2.1, *and $H$ is defined in terms of the $S$-operator $A_\mu$ as above then $\mu$ is equivalent to $\nu$ iff*

(i) $a_\mu - a_\nu \in H$

*and*

(ii) $(A_\nu x, y)_{\tilde{H}} = (A_\mu x, A_\mu y)_H - (KA_\mu x, A_\mu y)_H$

*for all $x, y \in \tilde{H}$ (and hence all $x \in B$) where $K$ is a symmetric Hilbert–Schmidt operator on $H$ with one not an eigenvalue of $K$.*

*Proof.* Since $\mu$ and $\nu$ are also Gaussian measures on $(\tilde{H}, \mathcal{O}\!\ell_{\tilde{H}})$ Theorem 4.7 of [7] implies that $\mu$ and $\nu$ are equivalent iff (i) holds and $\mu^*$ and $\nu^*$ are equivalent where $\mu^*$, $\nu^*$ have correlation operators $A_\mu$, $A_\nu$ and means zero. Furthermore, by (3) of Lemma 2.3 and an important result of Segal [13, p. 463] $\mu^*$ and $\nu^*$ are equivalent on $\tilde{H}$ (and hence on $B$) iff (ii) holds. That is, since $\mu^*$ is the restriction of the canonical normal distribution on $H$ to $\tilde{H}$ (Gross [8]) it follows from [13] and the reasoning used to prove (3) of Lemma 2.3 that $\mu^*$ is equivalent to $\nu^*$ iff for every $x$ in $\tilde{H}$ we have

$$(A_\nu x, x)_{\tilde{H}} = (TA_\mu x, TA_\mu x)_H \tag{3.1}$$

for some operator $T$ defined on $H$ such that $T^*T = I - K$ is non-singular on $H$ and $K$ is symmetric and Hilbert–Schmidt. In addition, it should be pointed out that the existence of the operator $T$ follows when $\mu^* \sim \nu^*$ since then the symmetric bilinear functional $\Omega(A_\mu x, A_\mu y) \equiv (A_\nu x, y)_{\tilde{H}}$ is bounded on $H$ and $T$ can be taken so that $\Omega(A_\mu x, A_\mu y) = (T^*TA_\mu x, A_\mu y)_H$. Now $T^*T = I - K$ nonsingular on $H$ and (3.1) implies that one is not an eigenvalue of $K$ and that (ii) holds. On the otherhand, if one is not an eigenvalue of $K$ and (ii) holds, then $T^*T = I - K$ is not only nonnegative but positive and hence $\mu^*$ is equivalent to $\nu^*$. This completes the proof.

## 4. Applications to Gaussian Processes

Let $\{x_t : 0 \leqslant t \leqslant T\}$ be a separable Gaussian process with covariance function $R(s, t)$ and mean $a(t)$. We will restrict our attention to mean-continuous processes which is equivalent to assuming that $R(s, t)$ is continuous on $[0, T] \times [0, T]$ and easily implies that $a(t)$ is continuous on $[0, T]$. Furthermore, we can and do assume that $\{x_t\}$ has its sample paths in $\mathscr{L}_2[0, T]$.

Since $R(s, t)$ is continuous and non-negative definite it has the eigenfunction expansion $\sum_n \lambda_n \phi_n(s) \phi_n(t)$ which converges uniformly on $[0, T] \times [0, T]$ [11, p. 245], the eigenvalues $\lambda_n$ are positive numbers such that $\sum_n \lambda_n < \infty$, and the eigenfunctions $\{\phi_n(t)\}$ are continuous orthonormal elements of $\mathscr{L}_2[0, T]$. Furthermore, $\{x_t\}$ induces a

Gaussian measure $\mu$ on the Hilbert space $\mathscr{L}_2[0, T]$ which is uniquely determined by the mean $a(t)$ and the operator

$$A_\mu x(t) = \int_0^T R(s, t)\, x(s)\, ds.$$

Using the notation of the previous sections we will denote $\mathscr{L}_2[0, T]$ by $B$ and since $B$ is a Hilbert space we simply define $\bar{H} = B$. Then the Hilbert space $H$ of Lemma 2.3 is the set of elements in $B$ which are in the span of the eigenfunctions $\{\phi_n(t)\}$ and such that $\sum_n (x, \phi_n)_B^2/\lambda_n$ is finite where, of course, $(x, y)_B = \int_0^T x(s)\, y(s)\, ds$. The inner product on $H$ is then

$$(x, y)_H = \sum_n \frac{(x, \phi_n)_B\, (y, \phi_n)_B}{\lambda_n}\,.$$

As a passing remark we mention that $H$ with inner product $(\ ,\ )_H$ is the reproducing kernel Hilbert space [1, p. 343] for the kernel $R(s, t)$. Consequently, Theorem 3.1 is related closely to the results of [9] and [10]. We state the following theorem without proof to indicate the similarity of Theorem 3.1 applied to Gaussian processes and the results of [9]. The notation $A \ll B$ for kernels $A, B$ on $[0, T] \times [0, T]$ means $B - A$ is nonnegative definite. The direct product $H \otimes H$ is as defined in [1, p. 359].

THEOREM 4.1. *If $\{y_t\}$ and $\{x_t\}$ are mean-continuous Gaussian processes on $[0, T]$ with covariance functions $R_1(s, t)$, $R(s, t)$ and mean functions $a_1(t)$, $a(t)$, respectively, then the measure $\nu$ induced on $\mathscr{L}_2[0, T]$ by $\{y_t\}$ is equivalent to the measure $\mu$ induced by $\{x_t\}$ iff*

(a) *$a(t) - a_1(t)$ is in $H$ and*

(b) *$R(s, t) - R_1(s, t) = J(s, t)$ where $mR \ll J \ll MR$ for constants $-\infty < m < M < 1$ and $J(s, t) \in H \otimes H$.*

LEMMA 4.2. *If $R(s, t) = \min(s, t)$ (the covariance function of the Wiener process) then the Hilbert space $H$ associated with $R(s, t)$ is identical with the Hilbert space of functions $x(t)$ on $\mathscr{L}_2[0, T]$ such that for $0 \leqslant t \leqslant T$ we have $x(t) = \int_0^t g(s)\, ds$ for $g$ in $\mathscr{L}_2[0, T]$ and $\| x \|_H^2 = \int_0^T [g(s)]^2\, ds$.*

*Proof.* Since $H$ consists of all functions $x(t)$ such that $\sum_j (x, \phi_j)_B^2/\lambda_j$

is finite where $\{\phi_j\}$ and $\{\lambda_j\}$ form an eigensystem for $R(s, t)$ it follows that $x$ is in $H$ iff

$$\| x \|_H^2 = \sum_{j=1}^{\infty} \left[\frac{(2j-1)\pi}{2T}\right]^2 \left[\int_0^T x(s) \sqrt{\frac{2}{T}} \sin \left[\frac{(2j-1)\pi}{2T} s\right] ds\right]^2 \quad (4.1)$$

is finite. Thus for all functions $x(t)$ such that $x(0) = 0$ and $x'(t)$ is square integrable we have

$$b_j = \frac{(2j-1)\pi}{2T} \int_0^T x(s) \sqrt{\frac{2}{T}} \sin \left[\frac{(2j-1)\pi}{2T} s\right] ds$$

$$= -\int_0^T x'(s) \sqrt{\frac{2}{T}} \cos \left[\frac{(2j-1)\pi}{2T} s\right] ds,$$

and hence $\int_0^T [x'(s)]^2 \, ds = \sum_{j=1}^{\infty} b_j^2 = \| x \|_H^2$ where the first equality holds because

$$\left\{\sqrt{\frac{2}{T}} \cos \left[\frac{(2j-1)\pi}{2T} s\right]\right\}$$

is a complete orthonormal set in $\mathscr{L}_2[0, T]$. Now by (4.1) it follows that if $x \in H \subseteq \mathscr{L}_2[0, T]$ and $c_j = b_j \cdot 2T/(2j-1)\pi$ then

$$\sum_{j=1}^{\infty} c_j \sqrt{\frac{2}{T}} \sin \left[\frac{(2j-1)\pi}{2T} s\right]$$

converges uniformly to $x(t)$ and the series

$$\sum_{j=1}^{\infty} b_j \sqrt{\frac{2}{T}} \left[\cos \frac{(2j-1)\pi}{2T} s\right]$$

is a function in $\mathscr{L}_2[0, T]$ whose integral over $[0, T]$ is $x(t)$, $0 \leqslant t \leqslant T$. Thus the lemma is proved.

THEOREM 4.2 (Shepp). *A mean-continuous Gaussian process $\{y_t\}$ on $[0, T]$ with covariance function $R_1(s, t)$ and mean $a_1(t)$ induces a measure $\nu$ on $B = \mathscr{L}_2[0, T]$ which is equivalent to the measure $\mu$ induced by the Wiener process iff there exists a symmetric square integrable kernel $\alpha(s, t)$ on $[0, T] \times [0, T]$ such that*

(1) *$a_1(t) = \int_0^t g(s) \, ds$ for $g$ in $\mathscr{L}_2[0, T]$, and*

(2) *$R_1(s, t) = \min(s, t) - \int_0^s \int_0^t \alpha(u, v) \, du \, dv$*

*where one is not an eigenvalue of the kernel $\alpha$.*

*Proof.* Since the Wiener process has mean zero and covariance $R(s, t) = \min(s, t)$ it follows from Lemma 4.2 that (1) is equivalent to (i) of Theorem 3.1. Let (2) hold and define $\Lambda(s, t) = \int_0^s \int_0^t \alpha(u, v)\, du\, dv$. Since $\alpha(u, v)$ is square integrable there exists an orthonormal set $\{\psi_j\}$ of eigenfunctions with eigenvalues $\{\mu_j\}$ such that $\sum_j \mu_j^2 < \infty$ and $\alpha(u, v) = \sum_j \mu_j \psi_j(u)\, \psi_j(v)$ where the series converges in the mean-square sense. Then

$$\Lambda(s, t) = \sum_j \mu_j \Gamma_j(s)\, \Gamma_j(t),$$

where $\Gamma_k(t) = \int_0^t \psi_k(u)\, du$ for $k = 1, 2, \ldots,\ 0 \leqslant t \leqslant T$ and the convergence is uniform in $(s, t)$. Since $\{\Gamma_k\}$ is an orthonormal set in $H$ and if we define

$$Ky(t) = \sum_j \mu_j (y, \Gamma_j)_H\, \Gamma_j(t),$$

then $K$ is a symmetric Hilbert–Schmidt operator on $H$ since $\sum_j \mu_j^2 < \infty$. Furthermore, for $x, y$ in $H$ we have

$$\begin{aligned} (Kx, y)_H &= \sum_j \mu_j (x, \Gamma_j)_H\, (y, \Gamma_j)_H \\ &= \sum_j \mu_j (x', \psi_j)_B\, (y', \psi_j)_B \\ &= \int_0^T \int_0^T \alpha(u, v)\, x'(u)\, y'(v)\, du\, dv. \end{aligned} \tag{4.2}$$

Thus one is not an eigenvalue of $K$ on $H$ iff one is not an eigenvalue of the kernel $\alpha$. Now easy calculations show that $(A_\mu x)'\,(t) = \int_t^T x(u)\, du$ for all $x$ in $B$ and hence by (4.2)

$$\begin{aligned} (KA_\mu x, A_\mu y)_H &= \int_0^T \int_0^T \alpha(s, t)(A_\mu x)'\,(s)(A_\mu y)'\,(t)\, ds\, dt \\ &= \int_0^T \int_0^T \Lambda(s, t)\, x(s)\, y(t)\, ds\, dt \end{aligned} \tag{4.3}$$

where the last equality follows by integration by parts. Combining (2) and (4.3) we have for all $x, y$ in $B$ that

$$\begin{aligned} (A_\nu x, y)_B &= (A_\mu x, y)_B - (KA_\mu x, A_\mu y)_H \\ &= (A_\mu x, A_\mu y)_H - (KA_\mu x, A_\mu y)_H \end{aligned}$$

where $K$ is a symmetric Hilbert–Schmidt operator on $H$ and one is not an eigenvalue of $K$ (combine (4.2) and (2)). Hence by Theorem 3.1 $\mu \sim \nu$.

Conversely, if $\nu$ is equivalent to $\mu$ then (1) holds by Lemma 4.2 and (i) of Theorem 3.1. To show (2) we define for $0 < t < T$ and $n$ sufficiently large the function $z_{n,t}(u)$ which is zero for $|t - u| > 1/2n$, non-negative and continuous on $[0, T]$, and such that $\lim_n \int_0^T z_{n,t}(u)\, du = 1$. Then

$$\lim_n (A_\mu z_{n,t}, z_{n,s})_B = \lim_n \int_0^T \int_0^T R(u, v)\, z_{n,t}(u)\, z_{n,s}(v)\, du\, dv$$
$$= R(s, t)$$

and $\lim_n (A_\nu z_{n,t}, z_{n,s})_B = R_1(s, t)$. Hence by (ii) of Theorem 3.1 we have

$$\lim_n (KA_\mu z_{n,t}, A_\mu z_{n,s})_H = R(s, t) - R_1(s, t). \tag{4.4}$$

Since $K$ is a symmetric Hilbert–Schmidt operator on $H$ with eigenvalues in $(-\infty, 1)$ it follows that for $y$ in $H$

$$Ky = \sum_j \mu_j (y, \Gamma_j)_H\, \Gamma_j$$

where $\{\Gamma_j\}$ is an orthonormal sequence in $H$ and $\sum_j \mu_j^2 < \infty$. Let $\psi_j(t) = \Gamma_j'(t)$ for $j = 1, 2, \ldots$. Then $\{\psi_j\}$ is an orthonormal set in $B$, and if

$$\alpha(s, t) = \sum_j \mu_j \psi_j(u)\, \psi_j(v)$$

then $\alpha(s, t)$ is symmetric and square integrable with eigenvalues $\{\mu_j\} \subseteq (-\infty, 1)$ and as in (4.3) for $x, y$ in $B$

$$(KA_\mu x, A_\mu y)_H = \int_0^T \int_0^T \Lambda(s, t)\, x(s)\, y(t)\, ds\, dt$$

where $\Lambda(s, t) = \int_o^s \int_o^t \alpha(u, v)\, du\, dv$. By (4.4) we then have

$$R(s, t) - R_1(s, t) = \lim_n (KA_\mu z_{n,t}, A_\mu z_{n,s})_H$$
$$= \int_0^T \int_0^T \Lambda(u, v)\, z_{n,t}(u), z_{n,s}(v)\, dv$$
$$= \Lambda(s, t) = \int_0^s \int_0^t \alpha(u, v)\, du$$

and the theorem is proved.

## 5. Gaussian Measures on the $\ell_p$ Spaces

Another application of Theorem 3.1 is to Gaussian measures on the Banach spaces $\ell_p$, $1 \leqslant p < \infty$, where by $\ell_p$ we mean the space of all real sequences $x = \{x_k\}$ such that $\| x \|_p = [\sum_{k=1}^{\infty} | x_k |^p]^{1/p}$ is finite. The definition of a Gaussian measure $\mu$ on $\ell_p$ is that given prior to Lemma 2.2. Now [15], [16] establish that $\mu$ is uniquely determined by an infinite symmetric nonnegative-definite matrix $S = (s_{ij})$ such that $\sum_{i=1}^{\infty} s_{ii}^{p\backslash 2} < \infty$ and a mean vector $a_\mu$ in $\ell_p$. Here $s_{ij} = \int_{\ell_p} x_i x_j \, d_\mu(x)$ and for $y$ in $\ell_p{}^*$ (the dual of $\ell_p$) $(y, a_\mu) = \int_{\ell_p} (y, x) \, d_\mu(x)$. We will call $S$ the $S_p$-operator associated with $\mu$.

Given a Gaussian measure $\mu$ on $\ell_p$ with $S_p$-operator $(s_{ij})$ we henceforth assume $(s_{ij})$ is diagonal and that $s_{ii} > 0$ for $i = 1, 2,\ldots$. Letting $\lambda = \{\lambda_i\}$ be an element in $\ell_{p/2}^*$ such that each $\lambda_i > 0$ we define the Hilbert space

$$\tilde{H} = \left\{ x = \{x_i\} : \sum_i \lambda_i x_i^2 < \infty \right\} \tag{5.1}$$

with the inner product

$$(x, y)_{\tilde{H}} = \sum_i \lambda_i x_i y_i \,. \tag{5.2}$$

Then $\ell_p \subseteq \tilde{H}$ and for $x \in \tilde{H}$ we define the operator $A_\mu$ as follows:

$$A_\mu(x) = \sum_i \lambda_i s_{ii} (x, e_i)_{\tilde{H}} \, e_i$$

where $e_i = (0,\ldots, \lambda_i^{-1/2}, 0,\ldots)$ $i = 1, 2,\ldots$. Then $\{e_i\}$ is an orthonormal basis for $\tilde{H}$, $(A_\mu x, x)_{\tilde{H}} = \sum_i \lambda_i^2 s_{ii} x_i^2 = \int_{\tilde{H}} (x, z)_{\tilde{H}}^2 \, d_\mu(z)$ since $\mu$ can also be considered as a measure on $\tilde{H}$, and if $\rho_i = \lambda_i s_{ii}$, $i = 1, 2,\ldots,$ then $\rho_i$ is the eigenvalue of $A_\mu$ corresponding to $e_i$.

The Hilbert space $H \subseteq \ell_p$ is

$$\begin{aligned} H &= \left\{ y = \{y_k\} : \sum_k \frac{1}{\rho_k} (y, e_k)_{\tilde{H}}^2 < \infty \right\} \\ &= \left\{ y = \{y_k\} : \sum_k y_k^2 / s_{kk} < \infty \right\} \end{aligned} \tag{5.3}$$

since $(y, e_k)_{\tilde{H}}{}^2 = \lambda_k y_k{}^2$, and

$$\begin{aligned} (x, y)_H &= \sum_k (y, e_k)_{\tilde{H}} (x, e_k)_{\tilde{H}} / \rho_k \\ &= \sum_k x_k y_k / s_{kk} \end{aligned} \tag{5.4}$$

is the inner product on $H$. Further, $\psi_i = (0,\ldots, \sqrt{s_{ii}}, 0,\ldots)$, $i = 1, 2,\ldots$, is an orthonormal basis for $H$.

The next lemma characterizes symmetric Hilbert–Schmidt operators on $H$ in terms of a matrix representation.

LEMMA 5.1. *If $K$ is a symmetric bounded linear operator on $H$ and $(k_{ij})$ is the real symmetric matrix such that for the orthonormal basis $\{\psi_i\}$ of $H$ we have*

$$\begin{aligned} K\psi_i &= (k_{1i}\sqrt{s_{ii}}, k_{2i}\sqrt{s_{ii}},\ldots) \\ &= \sum_j k_{ji}\frac{\sqrt{s_{ii}}}{\sqrt{s_{jj}}}\psi_j \end{aligned} \tag{5.5}$$

*then $K$ is a Hilbert–Schmidt operator iff*

$$\sum_i \sum_j k_{ij}^2 \frac{s_{ii}}{s_{jj}} < \infty. \tag{5.6}$$

*Proof.* By [6, p. 33–34] $K$ is Hilbert–Schmidt iff $\sum_i \| K\psi_i \|_H^2 < \infty$. However, $\| K\psi_i \|_H^2 = \sum_j k_{ji}^2(s_{ii}/s_{jj})$ thus $\sum_i \| K\psi_i \|_H^2 = \sum_i \sum_j k_{ji}^2(s_{ii}/s_{jj})$ and since $k_{ij} = k_{ji}$ $K$ is Hilbert–Schmidt iff (5.6) holds.

THEOREM 5.1. *Let $\mu$ and $\nu$ be Gaussian measures on $\ell_p$, $1 \leqslant p < \infty$, with means $a_\mu$, $a_\nu$ and $S_p$-operators $S = (s_{ij})$, $T = (t_{ij})$, respectively. If $S$ is diagonal and $\tilde{H}$ and $H$ are defined as in* (5.1) *and* (5.3) *then $\nu \sim \mu$ iff*

(1) $a_\mu - a_\nu \in H$ *and*

(2) $t_{ij} = s_{ij} - s_{ii}k_{ij}$ $(i, j = 1, 2,\ldots)$

*where $K = (k_{ij})$ is a Hilbert–Schmidt operator on $H$ defined from $(k_{ij})$ as in* (5.5) *and such that one is not an eigenvalue of $K$.*

*Proof.* If $\mu \sim \nu$ then by Theorem 3.1 we have that $a_\mu - a_\nu \in H$. Furthermore, by (ii) of Theorem 3.1 we have for $x, y \in \tilde{H}$

$$\begin{aligned} (A_\nu x, y)_{\tilde{H}} &= (A_\mu x, y)_{\tilde{H}} - (KA_\mu x, A_\mu y)_H \\ &= (A_\mu x, y)_{\tilde{H}} - (KA_\mu x, y)_{\tilde{H}}, \end{aligned} \tag{5.7}$$

where $K$ is a Hilbert–Schmidt operator on $H$ without one as an eigenvalue. Hence by Lemma 5.1 the matrix $(k_{ij})$ for $K$ (such that

(5.5) is satisfied) satisfies (5.6) and by choosing $x = \beta_i$, $y = \beta_j$ where $\beta_k = \lambda_k^{-1/2} e_k$ we have from (5.7) that

$$\begin{aligned} t_{ij} &= \int_{\ell_p} x_i x_j \, d\nu(x) = \int_{\tilde{H}} x_i x_j \, d\nu(x) \\ &= \int_{\tilde{H}} (x, \beta_i)_{\tilde{H}} (x, \beta_j)_{\tilde{H}} \, d\nu(x) \\ &= (A_\nu \beta_i, \beta_j)_{\tilde{H}} \\ &= (A_\mu \beta_i, \beta_j)_{\tilde{H}} - (K A_\mu \beta_i, \beta_j)_{\tilde{H}} \\ &= s_{ij} - s_{ii} k_{ij} \qquad (i, j = 1, 2, \ldots). \end{aligned} \tag{5.8}$$

Thus (2) holds.

On the otherhand, if (1) and (2) hold then by (2) and (5, 8) we have for $i, j = 1, 2, \ldots$ that

$$(A_\nu \beta_i, \beta_j)_{\tilde{H}} = (A_\mu \beta_i, \beta_j)_{\tilde{H}} - (K A_\mu \beta_i, \beta_j)_{\tilde{H}}, \tag{5.9}$$

where $K$ is Hilbert–Schmidt on $H$ and without one as an eigenvalue. Applying (5.9) to finite linear combinations $x, y$ of the $\beta_k$'s we obtain

$$\begin{aligned} (A_\nu x, y)_{\tilde{H}} &= (A_\mu x, y)_{\tilde{H}} - (K A_\mu x, y)_{\tilde{H}} \\ &= (A_\mu x, A_\mu y)_H - (K A_\mu x, A_\mu y)_H \end{aligned} \tag{5.10}$$

and by continuity (5.10) holds for all $x, y$ in $\tilde{H}$. Thus by (1) and (2) and Theorem 3.1 $\mu \sim \nu$.

The main defect of Theorem 5.1 is that the $S_p$-operator $S$ of $\mu$ is assumed to be diagonal. The following corollary of Theorem 5.1 indicates how one can replace certain $S_p$-operators $T = (t_{ij})$ by a diagonal $S_p$-operator which produces a Gaussian measure equivalent to the Gaussian measure induced by $T$. Thus the corollary partially eliminates the defect.

The symbol $\delta_{ij}$ denotes one if $i = j$ and zero if $i \neq j$.

COROLLARY 5.1. *If $T = (t_{ij})$ is the $S_p$-operator of the Gaussian measure $\nu$ with mean $a$ and if $S = (s_{ij})$ where $s_{ij} = \delta_{ij} t_{ij}$ then the Gaussian measure $\mu$ with mean $a$ and $S_p$-operator $S$ is equivalent to $\nu$ provided*

$$\sum_{i \neq j} \frac{(t_{ij})^2}{t_{ii} t_{jj}} < 1.$$

*Of course, we are assuming $t_{ii} \neq 0$ for $i = 1, 2, \ldots$.*

*Proof.* Let $H$ be defined as in (5.3) and let $k_{ij} = (\delta_{ij} - 1)\, t_{ij}/t_{ii}$ for $i, j = 1, 2, \ldots$ . Let $K = (k_{ij})$ be defined on $H$ as in Lemma 5.1. Then

$$\sum_{i,j} (k_{ij})^2 \frac{s_{ii}}{s_{jj}} = \sum_{i \neq j} \frac{t_{ij}^2}{t_{ii} t_{jj}} < 1$$

and hence $K$ is a symmetric Hilbert–Schmidt operator on $H$. Further,

$$\| K \|_H^2 \leqslant \sum_{i,j} (k_{ij})^2 \frac{s_{ii}}{s_{ij}}$$

so $\| K \|_H < 1$ and $K$ cannot have one as an eigenvalue. Further, for $i, j = 1, 2, \ldots$ we have

$$t_{ij} = s_{ij} - s_{ii} k_{ij}$$

so $\nu \sim \mu$.

As a final remark we mention that if $\nu_1, \nu_2, \mu$ are Gaussian measures on $\ell p$ with mean $a$ and $S_p$-operators $T_1 = (t_{ij}^{(1)})$, $T_2 = (t_{ij}^{(2)})$, $S = (s_{ij})$, respectively, and $\nu_1 \sim \nu_2 \sim \mu$ where $S$ is diagonal then for $i, j = 1, 2, \ldots$ we have

$$t_{ij}^{(1)} - t_{ij}^{(2)} = s_{ii}(k_{ij}^{(2)} - k_{ij}^{(1)})$$

where $K_1 = (k_{ij}^{(1)})$ and $K_2 = (k_{ij}^{(2)})$ are Hilbert–Schmidt operators on $H$ defined as in Lemma 5.1 not having one as an eigenvalue and

$$t_{ij}^{(n)} = s_{ij} - s_{ii} k_{ij}^{(n)} \qquad (i, j = 1, 2, \ldots; n = 1, 2).$$


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